Anomaly cancellations in brane tilings

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ABSTRACT: We re-interpret the anomaly cancellation conditions for the gauge symmetries and the baryonic flavor symmetries in quiver gauge theories realized by the brane tilings from the viewpoint of flux conservation on branes.

Contents

1. Introduction

The brane tilings [1–6], generalization of the brane box models [7–9], are powerful tools to construct $\mathcal{N} = 1$ chiral gauge theories. They visualize the structure of gauge theories realized on D3-branes probing toric Calabi-Yau cones as dimer diagrams on $\mathbf{T}^2$. Faces, edges and vertices in the diagrams correspond to the gauge groups, bi-fundamental matter fields and terms in the superpotentials, respectively. Roughly speaking, the dimer diagrams illustrate the structure of brane systems. Each face represents a stack of D5-branes of disk topology. If the number of the D5-branes on the face is N, SU(N) gauge theory arises on the disk. Edges in a dimer diagram represent an NS5-brane working as a frame supporting the D5-branes on the faces. However, the precise shape of the branes in the system was not clear until recent.

The "physical" meaning of the brane tilings is clarified in [5]. Let us consider D3-branes probing a toric Calabi-Yau cone $\mathcal{M}$. An $\mathcal{N} = 1$ gauge theory is realized on the D3-branes. The brane tiling realizing the same gauge theory is obtained in the following way. The toric diagram, consisting of vectors in three-dimensional integral lattice $\mathbf{Z}^3$, is often used to describe the structure of $\mathcal{M}$. The lattice corresponds to the toric fiber $\mathbf{T}^3$ of $\mathcal{M}$ and the vectors represent shrinking cycles in $\mathcal{M}$. Due to the Calabi-Yau condition, $c_1 = 0$, all the endpoints of the vectors lie on a two-dimensional sublattice in the $\mathbf{Z}^3$. This allows us to represent the toric diagram as a two-dimensional diagram. Taking T-duality along the $\mathbf{T}^2$ specified by the two-dimensional sublattice, we obtain an NS5-brane wrapped on a holomorphic curve Σ in $(\mathbf{C}^*)^2$ [2]. The curve Σ is described by the Newton's polynomial associated with the toric diagram. The D3-branes in $\mathcal{M}$ is mapped to D5-branes wrapped on the dual $\mathbf{T}^2$. These D5-branes split into disks with their boundaries on the NS5-brane Σ. This system, consisting of the NS5-brane and the D5-branes, are nothing but the brane tiling.

By performing another T-duality transformation along a direction perpendicular to the $(\mathbf{C}^*)^2$, in which the NS5-brane worldvolume Σ is embedded, we obtain a non-compact Calabi-Yau manifold $\mathcal{W}$, which is mirror to $\mathcal{M}$. The D5-branes are transformed to D6-branes wrapped on 3-cycles in $\mathcal{W}$. The manifold $\mathcal{W}$ can be represented as $\mathbf{C}^*$ fibration over $(\mathbf{C}^*)^2$, and Σ describes the singular locus in $(\mathbf{C}^*)^2$ on which the $\mathbf{C}^*$ fiber is pinched. In Ref. [5], a simple procedure, referred to as "untwisting", is proposed, which determines the topology of Σ and the cycles wrapped by the D6-branes from dimer diagrams.

In this paper, we use the brane tiling picture, while Ref. [5] mainly adopts the picture with the mirror Calabi-Yau $\mathcal{W}$. Because these two pictures, as mentioned in [5], are directly related by T-duality, we can use the untwisting procedure as it is to determine the NS5-brane worldvolume Σ and the boundaries of D5-branes on Σ.

Let us briefly review the untwisting procedure [5]. Given a dimer diagram, we first make holes on all the faces leaving narrow region around the edges. As a result, we have a surface made of strips connected at the vertices. In order to obtain the worldvolume Σ of the NS5-brane, we twist all the strips by 180 degrees. Because the graph is bipartite [2], this procedure transforms the torus with holes to another orientable surface with holes. This surface is identified with the worldvolume Σ of the NS5-branes. The closed path consisting of the edges belonging to a face a in the dimer diagram becomes a zig-zag path [4] on Σ. This path is identified with the boundary of the D5-branes corresponding to the face a.

Because we now know (at least the topologies of) the shapes of branes in brane tilings, we can use them not simply as diagrams encoding the information of gauge theories, but as dynamical brane systems. In this paper, we investigate the cancellation of two kinds of anomalies from this viewpoint. The two gauge fields on the NS5-brane, the non-dynamical zero-form flux and the U(1) vector field, play important roles.

In the next section, we investigate the relation between zero-form flux conservation on the NS5-brane and the gauge anomaly cancellation. We show for general brane tilings that the anomaly cancellation for the gauge symmetries follows from the conservation of the zero-form flux on the NS5-brane worldvolume.

In section 3, after reviewing a similarity of the anomaly cancellation conditions for baryonic U(1) symmetries with those for the gauge symmetries, we show that a certain relation obtained from the boundary condition for the U(1) gauge fields on NS5 and D5-branes guarantees the cancellation of the anomaly associated with the baryonic U(1) symmetries.

2. Gauge anomaly cancellation

Let us consider the quiver gauge theory described by a brane tiling. Let n_G denote the number of gauge groups and $N(a)$ $(a = 1, \ldots, n_G)$ be the number of colors of the a-th gauge group. If they are all the same, the gauge anomalies cancel. In this section we do not impose this condition. Because brane tilings in general give chiral gauge theories, we cannot take arbitrary sets of ranks. The gauge anomaly cancellation requires $N(a)$ satisfy certain conditions. It is known that the number of independent ranks of gauge groups which satisfy the anomaly cancellation conditions is $d - 2$ where d is the perimeter of the

toric diagram associated with the Calabi-Yau cone $\mathcal{M}$ dual to the brane tiling [15, 16]. Furthermore, it is well known that the number of gauge groups, n_G, is twice the area of the toric diagram [1, 2]. Therefore, the number of the conditions imposed on $N(a)$ is

$$n_G - (d - 2) = 2n_{\text{internal}}, \tag{2.1}$$

where n_{internal} is the number of internal points of the toric diagram.

What are the corresponding conditions in the brane picture? Although it has been anticipated since the brane box models [7–9] were proposed that the conditions arise from the flux conservation on branes, and there are works relating the gauge anomaly cancellation in brane box models to the flux conservation and brane bendings [10–14], it has not been explicitly shown for general brane tilings due to the lack of the precise description of tilings in terms of physical branes. In this section, we clarify the relation between the anomaly cancellation and the zero-form flux conservation on the NS5-brane by using the untwisting procedure proposed in [5].

As we mentioned above, each gauge group lives on a stack of D5-branes of disk topology, and the boundary of the disk is on the NS5-brane Σ. Let us denote the disk corresponding to the gauge group $\text{SU}(N(a))$ by D_a, and its boundary by C_a. We can represent the D5-brane worldvolume as the formal sum

$$\sum_{a=1}^{n_G} N(a) D_a. \tag{2.2}$$

On NS5-branes, there exist two kinds of gauge fields. One is the U(1) vector gauge field B with 2-form flux, and the other is the non-dynamical zero-form flux p. Because the flux p couples to the "wall charge" of boundaries of D5-branes, the following equation holds in Σ:

$$\sum_{a=1}^{n_G} N(a) \delta(C_a) = dp, \tag{2.3}$$

where $\delta(C_a)$ is the delta function with support C_a. If $p \neq 0$ on the NS5-brane, the brane is no longer a pure NS5-brane but is a $(p, 1)$ 5-brane.[1] However, we refer to it as NS5-brane in what follows for simplicity. The curve Σ, given by Newton's polynomial associated with the toric diagram, is a surface with some punctures. Although it is a non-compact manifold, the total wall charge in it must vanish for the flux to conserve just like in the case of a compact manifold because the punctures cannot be sources nor sinks for the flux p. This means the following homology relation in the closure $\overline{\Sigma}$, which is obtained by closing the punctures in Σ, must hold:

$$\sum_{a=1}^{n_G} N(a) C_a = 0. \tag{2.4}$$

Note that the number of independent conditions given by (2.4) is the same with (2.1). This is because the genus of $\overline{\Sigma}$ described by the Newton's polynomial of the toric diagram is the same with the number of the internal points n_{internal} in the toric diagram, and the number

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[1]The value of the flux p at each puncture on Σ represents the D5-brane charge of the corresponding external line in the web diagram. These numbers assigned to the external lines are the same with the parameters b_i introduced in [16].

of the independent cycles on $\overline{\Sigma}$ is twice the genus. The fact that (2.4) gives the same number of conditions with those derived from the requirement of the anomaly cancellation strongly suggests that these two sets of conditions are in fact the same.

To show this is true, we need to treat objects in dimer diagrams. For this purpose, we first define some notations. We use the following indices for faces, edges and vertices in dimer diagrams:

$$a, b, \ldots : \text{faces}, \quad I, J, \ldots : \text{edges}, \quad i, j, \ldots : \text{vertices}. \tag{2.5}$$

Because faces, edges and vertices correspond to gauge groups, bi-fundamental matter fields and terms in the superpotential, respectively, we use indices in (2.5) to label these objects in the quiver gauge theories, too.

We use the character $\in$ to represents the adjacency. For example, $\sum_{I \in a}$ means the summation over all the edges adjacent to the face a.

If a is a face and I is an edge belonging to a, $I \cdot a$ means the face which is adjacent to a by the edge I. If $b = I \cdot a$, the relation $a = I \cdot b$ also holds.

The dimer diagrams are bipartite [2], namely, all the vertices can be colored with two colors, black and white, in such a way that the colors of two end points of an edge are always different. Thus we can define the orientation of edges, e.g., from white to black vertices.

In order to represent the relation of faces and edges, we define a function $\sigma(I, a)$. This function, defined for a face a and its side I, gives $+1$ if the orientation of the edge I is counter-clockwise as a side of the polygon a. Otherwise, this function gives -1.

We also define a function $\sigma(a, b)$ for two adjacent faces a and b by $\sigma(a, b) = \sigma(I, b) = -\sigma(I, a)$, where I is the edge shared by a and b. If the two faces share more than one edge, we need to specify an edge to define this function. This function represents the chirality of the bi-fundamental field corresponding to the edge I. If $\sigma(a, b) = +1$, the bi-fundamental field belongs to $(\overline{N}(a), N(b))$ of the $SU(N(a)) \times SU(N(b))$, and if $\sigma(a, b) = -1$, the field belongs to $(N(a), \overline{N}(b))$.

Let us represent the anomaly cancellation conditions in terms of these notations. The bi-fundamental matter fields coupling to $SU(N(a))$ are represented as edges $I \in a$. If $\sigma(I, a)$ is $+1$ (-1), the field belongs to the fundamental (anti-fundamental) representation of $SU(N(a))$, and the number of the flavours is the size $N(I \cdot a)$ of the gauge group for the adjacent face $I \cdot a$. Thus, we can write the gauge anomaly cancellation conditions as

$$\sum_{I \in a} \sigma(I, a) N(I \cdot a) = 0, \qquad \forall a, \tag{2.6}$$

or equivalently as

$$\sum_{b \in a} \sigma(a, b) N(b) = 0, \qquad \forall a. \tag{2.7}$$

As we mentioned above, a and b in (2.7) may share more than one edge. In such a case, we have to sum the contribution of all the edges shared by a and b separately.

In order to associate these conditions to the flux conservation on the NS5-brane Σ, we need to re-interpret the conditions from the viewpoint of the curve Σ. By the untwisting procedure, the edges belonging to a face a is mapped to a zig-zag path on Σ. This path

is identified with the boundary of D5-branes, C_a. An edge I is always shared by two faces on $\mathbf{T}^2$. This means that the edge I on Σ can be regarded as an intersection of two zig-zag paths on Σ. Because zig-zag paths are boundaries of D5-branes, open strings can stretch between two cycles at the intersection. From these open strings the chiral multiplet corresponding to the edge arises [17].

The function $\sigma(a,b)$ is identified with the intersection number $\langle C_a, C_b \rangle$ of the two paths C_a and C_b on Σ. As we mentioned above, two faces on $\mathbf{T}^2$ can meet on more than one edge. In such a case, corresponding paths C_a and C_b also have more than one intersecting point and $\sigma(a,b)$ (with I implicitly specified) gives the intersection number of one of the intersecting points. In terms of the intersection, we can rewrite the condition (2.7) as

$$\langle C_a, \sum_{b=1}^{n_G} N(b) C_b \rangle = 0, \qquad \forall a. \tag{2.8}$$

This relation automatically holds if the wall charge cancellation condition (2.4) is satisfied. The converse is also correct; Because we know that the number of the independent conditions is the same with the number of the independent cycles in Σ, the condition (2.8) is equivalent to the condition (2.4).

3. Baryonic U(1) symmetries

In this section, we consider baryonic flavor symmetries in superconformal quiver gauge theories with all the gauge groups having equal rank. It is known that there is a formal similarity between the gauge anomaly cancellation and the baryonic U(1) anomaly cancellation [15, 16]. Because of this, we expect that the cancellation of the baryonic U(1) symmetries is also explained by a similar mechanism associated with flux conservation on Σ. Let us first review the similarity.

To see this similarity, it is convenient to introduce the "gradient" $\beta = dN$ of the sizes of the gauge groups $N(a)$. (Although we assume that all the sizes $N(a)$ of the gauge groups are equal when we consider the baryonic U(1) symmetries below, we first consider the gauge anomalies again, which arises when $N(a)$ depends on a, in order to look at the similarity between them.) The function β gives the assignment of numbers for edges and explicit definition of the gradient is given by

$$\beta(I) = \sum_{b \in I} \sigma(I, b) N(b). \tag{3.1}$$

Using the identity $\sum_{I \in a} \sigma(I, a) = 0$, which holds in bipartite graphs, we can show that the anomaly cancellation condition (2.6) is equivalent to

$$\sum_{I \in a} \beta(I) = 0, \qquad \forall a. \tag{3.2}$$

The reason we introduced the function $\beta(I)$ is that this function satisfies the same conditions with those imposed on the charge assignment of the anomaly-free baryonic U(1) symmetries. Let $Q(I)$ denote the charge assignment of a baryonic U(1) symmetry. This must satisfy the curl-less condition $dQ = 0$, or equivalently,

$$\sum_{I \in i} Q(I) = 0, \qquad \forall i \tag{3.3}$$

to keep the superpotential invariant. Any U(1) flavor symmetry must satisfy (3.3). In addition to this, the baryonic symmetries must satisfy the following condition:

$$\oint_\alpha Q = \oint_\beta Q = 0, \tag{3.4}$$

where α and β are paths consisting of faces and edges, and belonging to the two independent homology classes of the torus. The integral along a path is defined as follows: If a closed path μ is given as a periodic sequence $\mu = \{\ldots, I_{n-1}, a_{n-1}, I_n, a_n, I_{a+1}, a_{n+1}, \ldots\}$ whose two adjacent components (e.g. the face a_{n-1} and the edge I_n) are also adjacent in the dimer diagram, the integral along μ is defined by

$$\oint_\mu Q \equiv \sum_n \sigma(I_n, a_n) Q(I_n). \tag{3.5}$$

The condition (3.4) guarantees that any mesonic chiral operators decouple from the symmetry.

The condition (3.3) and (3.4) are nothing but the integrability condition which guarantees the existence of $S(a)$ satisfying

$$Q = dS, \tag{3.6}$$

where the gradient dS is defined in the same way with (3.1). This relation is identical with the relation $\beta = dN$ for the gauge anomaly cancellation. Furthermore, the cancellation of the $\mathrm{SU}(N(a))^2\, \mathrm{U}(1)_{\mathrm{baryonic}}$ triangle anomalies requires $Q(I)$ satisfy

$$\sum_{I \in a} Q(I) = 0, \qquad \forall a. \tag{3.7}$$

This is identical to the condition (3.2) for the gauge anomaly cancellation.

From this similarity, we can conclude that there exists almost one to one correspondence between the baryonic U(1) symmetries and the rank distributions of anomaly-free gauge groups. Note that if all sizes of the gauge groups are the same, the divergence dN vanishes and there is no corresponding baryonic symmetry. Therefore, the number of the independent $N(a)$ is greater than the number of the baryonic symmetries by one. In other words, there is one-to-one correspondence between fractional branes and anomaly-free baryonic flavor symmetries [15, 16]. It is known that the number of the baryonic U(1) symmetries is $d - 3$, where d is the perimeter of the toric diagram of the dual Calabi-Yau cone $\mathcal{M}$ [3].

Using this analogy, we can immediately rewrite the anomaly freedom condition (3.7) as the following homology relation in $\overline{\Sigma}$:

$$\sum_{a=1}^{n_G} S(a)C_a = 0. \tag{3.8}$$

Formally, this is obtained by replacing $N(a)$ in (2.4) by $S(a)$. What is the interpretation of this condition in the brane picture? To answer this question, we should identify the (non-dynamical) gauge fields associated with the symmetries.

A baryon is the bound state of N bi-fundamental fields. In the brane picture, it is a bound state of N open fundamental strings stretched between two D5-branes. This implies that the gauge fields couples to the baryons are the diagonal U(1) gauge fields on D5-branes.

On the other hand, in the AdS dual picture, baryons are realized as D3-branes wrapped on 3-cycles in $\mathcal{M}$, and by taking T-duality transformation to go to the brane picture, the wrapped D3-branes are mapped to D-strings with their ends on the NS5-brane. Thus, the gauge field coupling to the baryons seems to be the U(1) gauge field on the NS5-brane, which couples to the endpoints of D-strings.

We now have two kinds of gauge fields as candidates for the gauge fields coupling to baryons. In fact, the correct answer is the combination of these two. The important fact is that the U(1) gauge fields on the D5-branes and the NS5-brane are correlated with each other via the boundary condition on the cycles C_a. Let A^a be the diagonal U(1) gauge field on D_a and B the gauge field on Σ. The boundary condition imposed on them on the boundary C_a is

$$(A^a - \Delta B)|_{C_a} = 0, \tag{3.9}$$

where ΔB means the discontinuity of B, and $|_{C_a}$ means the pull-back on C_a. An easy way to obtain this boundary condition is to transform the system to an NS5-D4 system by T-duality, and lift it to a smooth M5-brane configuration. The boundary condition (3.9) is obtained from the continuity of the two-form gauge field on the M5-brane. If we denote the four-dimensional (non-dynamical) gauge field for a baryonic symmetry by $\mathcal{A}$, the gauge fields on branes are decomposed as

$$A^a = s^a \mathcal{A}, \qquad B = f \mathcal{A} \tag{3.10}$$

where s^a are function on the disks D^a, and f is a function on the curve Σ. These functions depend only on the coordinates of the internal space, while $\mathcal{A}$ is a 1-form depending on the coordinates in the four-dimensional spacetime. For the symmetry not to be broken, the gauge field $\mathcal{A}$ must be massless, and this implies that the functions s^a and f must be constant except on the boundaries C_a. The constants s^a assigned to D_a should be identified with the function $S(a)$ in (3.6). Because a bi-fundamental field arises from open strings at the intersection of two cycles C_a and C_b on Σ, its charge for the baryonic U(1) symmetry is given as the difference of $S(a)$ and $S(b)$. This is the physical interpretation of the relation (3.6). By substituting (3.10) into (3.9), we obtain the following equation for

the function f:

$$\sum_{a=1}^{n_G} S(a)\delta(C_a) = df. \tag{3.11}$$

Thus we can treat $S(a)$ as wall charges carried by C_a, and f as a zero-form flux coupling to the charges $S(a)$. This relation is formally identical to that between the wall charges of D5-brane boundaries and the zero form flux p discussed in the previous section. For the flux conservation, the total charge must vanish;

$$\sum_{a=1}^{n_G} S(a)C_a = 0. \tag{3.12}$$

Just like in the case of the gauge anomalies, this condition guarantees the cancellation of the anomalies of the baryonic U(1) symmetries.

4. Conclusions

By using the untwisted procedure proposed in [5], which relates a dimer diagram with the holomorphic curve Σ representing the NS5-brane in the corresponding brane tiling, we showed that the anomaly cancellations of the gauge symmetries and the baryonic flavor symmetries are explained by considering the behavior of two kinds of gauge fields on the NS5-brane.

The zero-form flux conservation on the NS5-brane requires cancellation of the wall charge of the boundaries of the D5-branes. This imposes conditions on the numbers of D5-branes on the faces of the dimer diagram, and these conditions guarantee the gauge anomaly cancellation.

We identified the gauge fields coupling to the baryonic U(1) symmetries with combinations of the vector fields on the D5-branes and the NS5-brane in the brane tiling. The correlation of these vector fields at the junctions of the NS5 and D5-branes imposes conditions on the baryonic U(1) charge assignment for the bi-fundamental fields. These conditions guarantee the cancellation of the anomaly of the baryonic U(1) symmetries.

Acknowledgments

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